Consistent dynamic choice and non-expected utility preferences

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Abstract

This paper studies the application of the two most popular non-expected utility (NEU) models -Choquet Expected Utility (CEU) and Maximin Expected Utility (CEU)- to dynamic choice situations in a purely subjective framework. We give an appropriate version of the reduction of compound acts axiom, that states the equivalence between a static and a dynamic choice situation. We show that if consequentialism -only those consequences that can be reached do matter- is additionally assumed, then a monotonic constant linear representation degenerate into expected utility. We envisage two different ways to resolve this problem for the cases where the representation is a CEU or a MEU one. One way consists to weaken the reduction of compound acts axiom, which does not hold on all events. Another way is to relax consequentialism. Then we axiomatically characterize an updating rule for both approaches allowing recursion in several cases.

Key-words: Choquet expected utility; Maximin expected utility; Consequentialism; Reduction of compound acts; Dynamic choice; Updating

JEL classification numbers: D 81, D 83

1 Introduction

In decision theory under uncertainty, typical violations of the expected utility (EU) paradigm have lead to the development of Non-expected utility (NEU) criteria. In this paper, we will focus on the most popular approaches, namely the Choquet Expected Utility (CEU) model and the Maximin Expected Utility (MEU) model. These models, which do not assume that the decision maker's beliefs are represented by a single additive prior, have been successfully applied to various economics situations. However, a wide class of situations involves sequential resolution of the uncertainty, and the use of NEU models in this context is a major concern.

Since seminal works of Hammond (1988), Machina (1989), Segal (1990) and Karni and Schmeidler (1991) in a risky setting, it is well-known that NEU models cannot

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accommodate a backward induction procedure, which is often seen as the conjunction of three different principles:

- Dynamic consistency: the decision maker's preferences are unchanged throughout the decision tree;
- Consequentialism: the decision maker's preferences are only dependent on the information received. In particular, counterfactuals outcomes are not relevant;
- Reduction of compound lotteries: the decision maker is indifferent between a static and a dynamic choice situation, provided the final consequences be the same. In other words, lotteries are reduced by probability calculus.

Together with standards axioms, these three principles imply expected utility in choice under risk. In settings of knightian uncertainty, where probability distributions on the outcomes are not given, relations among these axioms is unclear. The main reason is that the objects of choice are Savage acts, which are maps from the state space into the set of final consequences. It does not allow to perform an intuitive formalization of the reduction of compound lotteries axiom in a purely subjective set-up.

That is why previous works in the literature often link the first and the third principle into a same axiom, called dynamic consistency in Epstein and Le Breton (1993), Sarin and Wakker (1998), Ghirardato (2002), Epstein and Schneider (2003), Eichberger et al. (2005) and Dominiak and Lefort (2009).

However, these works differ, depending on the algebra of events. In particular, if the information structure is represented by a given and fixed filtration, then expected utility can be avoided. This paper first aims at showing that the differences between the results in the existing literature can be axiomatically explained. Precisely, we show that i) consequentialism and reduction of compound acts together with a monotonic constant linear representation imply expected utility and ii) if the decision maker's preferences are represented by a CEU or a MEU form, a weakening of reduction of compound acts allow to avoid expected utility in the same way than Sarin and Wakker (1998) and Epstein and Schneider (2003). Concerning CEU preferences, a necessary and sufficient condition to weak reduction of compound acts and consequentialism hold together is that the capacity be additive on a part of the algebra of events. In the MEU framework, the set of priors must be rectangular.

These conditions constitute strong restrictions on NEU criteria, and lead us to relax the consequentialism assumption. Then the main difficulty is to obtain well-defined conditional preferences in order to use a backward induction procedure. This goal can be reached in several situations, for which we axiomatically characterize an updating rule allowing a recursive representation of the preferences. These situations are the same for the two approaches (CEU and MEU), which coincide when the set of priors is the core of a convex capacity.

The remainder of the paper is organized as follows. In the next section we introduce notations and axioms, and we show that a monotonic constant linear representation cannot accommodate both reduction of compound acts and consequentialism. Moreover, we present our updating rule in an example. Section 3 and 4 report results for CEU and MEU preferences, respectively. Section 5 concludes.

2 Set-up, axioms and motivations

We consider a finite state space S such that $|S| = N \geq 3$. A state in S is described by s. An event is a subset of S and for all $B \subset S$ we note B^c the event $S \backslash B$. We note $\Sigma = 2^S$ the algebra of S. The set of outcomes is X such that $X = [x_*; x^*] \subset R$, with $x^* > x_*$. The set of simple acts (measurable functions taking only finite values) is noted $\mathcal{A} \subseteq X^N = \{f : S \longrightarrow X\}$. We identify X with the subset of the constant acts in $\mathcal{A}$. We write $f =_E g$ if $f(s) = g(s)$ for all s in E and $f = x_E y$ denotes the binary act yielding $f(s) = x$ when $s \in E$ and $f(s) = y$ when $s \in E^c$.

A decision maker (DM for short) is characterized by a class of binary relations $\{\succcurlyeq_E\}_{E \in \Sigma}$ on $\mathcal{A}$. For all E in Σ, $\succcurlyeq_E$ compares acts when she is informed that the right state is in E. When $E = S$, $\succcurlyeq_S$ is noted $\succcurlyeq$ and compares acts when no information is given to the DM.

First assume that $\{\succcurlyeq_E\}_{E \in \Sigma}$ can all be represented by monotonic constant-linear functionals with certainty equivalents, as axiomatized by Ghirardato and Marinacci (2001) and Ghirardato et al. (2001, 2003) in a static framework. Such a representation of the DM's preference generalizes the NEU models examined in this paper, in the sense that it constitutes the weakest model achieving a separation of cardinal utility and a unique representation of beliefs. Separation of utility and beliefs means that tastes are constant whereas beliefs can change with new information. For any E in Σ, we note $I_E[u(.)]$ the criterion representing $\succcurlyeq_E$ so that $f \succcurlyeq_E g$ if and only if $I_E[u(f)] \geq I_E[u(g)]$ for all f and g in $\mathcal{A}$. When $E = S$, we note $I[u(.)]$ the representation of $\succcurlyeq$. The only requirements on $\{I_E\}_{E \in \Sigma}$ are (i) monotonicity, i.e. $f \geq g$ implies $I_E(f) \geq I_E(g)$ for all $f, g \in \mathcal{A}$, and (ii) constant linearity, i.e. $I_E(af + b) = aI_E(f) + b$ for all $f \in \mathcal{A}, a \in R_+$ and $b \in R$. The utility function $u : X \longrightarrow R$ represents $\{\succcurlyeq_E\}_{E \in \Sigma}$ on X, and it is unique up to a positive affine transformation.

Therefore, for any f, $I_E[u(f)] = I_E[u(f(s_1)), ..., u(f(s_N))]$. Let x and y in X be certainty equivalents of f given E and E^c respectively, such that $x \sim_E f$ and $y \sim_{E^c} f$. Then $I_E[u(f)] = u(x)$ and $I_{E^c}[u(f)] = u(y)$.

A common assumption in risky situations is that the DM reduces a dynamic decision problem to a static one by probability calculus. Such an assumption is named "Reduction of compound lotteries"(RCL). In situations of knightian uncertainty, the set-up is necessarily static when the objects of choice are acts, that are maps into the set of *final* outcomes. Indeed, compound acts (or strategies, i.e. a complete set of moves through the sequential decision tree), that pay another act (a sub-act) are not the objects of choice in the Savage framework. Therefore, in order to formalize a reduction of compound acts assumption, assume that the DM has to make a choice between the act f and the binary act $x_E y$. The former describes a static situation whereas the latter depicts a dynamic situation, because x and y are recursively calculated. It may happen that the DM be not indifferent between a sequential resolution of the uncertainty and a one-shot one. When probabilities are not known, a similar assumption to RCL is that the DM be indifferent between $x_E y$ and f, i.e. she reduces a dynamic situation to a static one.

Axiom 1 *(Reduction of Compound Acts) For all E in Σ, f in $\mathcal{A}$ and x and y in X, if $f \sim_E x$ and $f \sim_{E^c} y$ then $f \sim x_E y$.*

It implies $I[u(f)] = I[u(x_E y)]$, hence the following recursive relation

$$I[u(f)] = I[I_E[u(f)], ..., I_E[u(f)], I_{E^c}[u(f)], ..., I_{E^c}[u(f)]] \quad (1)$$

holds true and, with a slight abuse of notations, $I[u(f)] = I[I_E[u(f)], I_{E^c}[u(f)]]$. It allows to resolve a dynamic decision problem by backward induction. Moreover, the following property is satisfied.

Property 1 *(Dynamic Consistency) For all E in Σ and f and g in $\mathcal{A}$, $f \succcurlyeq_E g$ and $f \succcurlyeq_{E^c} g$ imply $f \succcurlyeq g$.*

Indeed, $f \succcurlyeq_E g$ and $f \succcurlyeq_{E^c} g$ if and only if (by transitivity) $x \succcurlyeq_E x'$ and $y \succcurlyeq_{E^c} y'$, where $x \sim_E f, x' \sim_E g, y \sim_{E^c} f$ and $y' \sim_{E^c} g$. By monotonicity, $x_E y \succcurlyeq x'_E y'$ and then RCA implies $f \succcurlyeq g$.

It is worth noting that such a version of dynamic consistency is specific to Savage acts. Indeed, when the decision maker's preferences are defined on strategies, or compound lotteries[1], dynamic consistency is defined differently. For instance, let $N = 4$, $p(\{s_1, s_2\}) = p_1$, $p(\{s_3, s_4\}) = p_2$ and let $l = (l_1, p_1; l_2, p_2)$ be a compound lottery yielding sub-lotteries $l_1 = (x_1, q_1; x_2, q_2)$ and $l_2 = (x_3, q_3; x_4, q_4)$. Then dynamic consistency implies that if $l_1 \succcurlyeq_{\{s_1,s_2\}} l'_1$ and $l_2 \succcurlyeq_{\{s_3,s_4\}} l'_2$ then $l \succcurlyeq (l'_1, p_1; l'_2, p_2) = l'$. In other words, it means that if the continuation of the strategy l is preferred to the continuation of l' whatever new information learned, then the strategy l is preferred to the strategy l'.

In a purely subjective set-up, a similar definition should be: if $f \succcurlyeq_E g$ and $f \succcurlyeq_{E^c} g$ then $x_E y \succcurlyeq x'_E y'$, with x, y, x' and y' defined as above. Therefore, axiom 1 does not imply itself property 1, and an additional dynamic extension of monotonicity, defined as dynamic consistency in a risky setting, is needed.

For convenience when updating, the following assumption, broadly discussed in Machina (1989), is generally made in dynamic models of choice under uncertainty:

Axiom 2 *(Consequentialism) For all f and g in $\mathcal{A}$ and E in Σ, $f =_E g$ implies $f \sim_E g$.*

Let x and y in X such that $x > y$. W.l.o.g. we normalize the utility function so that $u(x) = 1$ and $u(y) = 0$. Then, for any A in Σ, $I[u(x_A y)] = I(1_A)$, where 1_A is the characteristic function of event A, denotes the individual evaluation of the likelihood of A. The conditional evaluation of event A given event E is noted $I_E(1_A)$.

Proposition 1 *Let $\{\succcurlyeq_E\}_{E\in\Sigma}$ be a class of preference relations on $\mathcal{A}$. Then the following two statements are equivalent:*

i $\{\succcurlyeq_E\}_{E\in\Sigma}$ satisfy axioms 1 and 2 and it is represented by the class of monotonic constant linear functionals $\{I_E[u(.)]\}_{E\in\Sigma}$;

ii There exists a unique probability measure $p : \Sigma \to [0, 1]$ such that $\succcurlyeq$ is represented by

$$I : f \longmapsto \int_S u(f) dp \quad (2)$$

[1] As in Kreps and Porteus (1978), Karni and Schmeidler (1991) and Volij (1994).

Moreover, each conditional evaluation $\{I_E[u(.)]\}_{E\in\Sigma}$ *is also an expected utility functional using the bayesian updating of* p*, denoted* p_E*, and the utility function* $u(.)$.

Proof $(ii) \implies (i)$ is straightforward. We prove $(i) \implies (ii)$. Let $p : \Sigma \to R$ be a set function such that $I(1_A) = p(A)$ for all A in Σ. $p_E(A) = I_E(1_A)$ is defined similarly. Then p satisfies the following properties: $p(\emptyset) = I(1_\emptyset) = 0, p(S) = I(1_S) = 1, \forall A, B \in \Sigma, A \subseteq B \Rightarrow I(1_A) \leq I(1_B) \Rightarrow p(A) \leq p(B)$. Therefore, p is a capacity across binary acts[2]. We have to prove that p satisfies the additional following property: $p(A) + p(B) = p(A \cup B)$ for all disjoints A and B. In this case, p is finitely additive and it is a probability. By axiom 1, for all $E \in \Sigma$ and $A \subset E$,

$$I(1_A) = I[I_E(1_A)] \tag{3}$$

hence $I(1_A) = I(p_E(A), 0) = p(E) \cdot p_E(A)$. Moreover, by axiom 2, for all B in Σ such that $B \cap E = A$, $I_E(1_B) = I_E(1_A)$ and then

$$p_E(A) = \frac{p(A \cap E)}{p(E)} \tag{4}$$

for all A and E in Σ. Let $S = (A \cup B \cup C)$, with $A \cap B = B \cap C = A \cap C = \emptyset$. With relation (4),

- if $E = (A \cup B)$,

$$I(1_{A\cup C}) = I[I_{A\cup B}(1_{A\cup C}), I_C(1_{A\cup C})] = I[\frac{p(A)}{p(A\cup B)}, 1] = p(C) + [1-p(C)] \cdot \frac{p(A)}{p(A\cup B)} \tag{5}$$

$$I(1_{B\cup C}) = p(C) + [1 - p(C)] \cdot \frac{p(B)}{p(A\cup B)} \tag{6}$$

- if $E = (A \cup C)$,

$$I(1_{A\cup B}) = p(B) + [1 - p(B)] \cdot \frac{p(A)}{p(A\cup C)} \tag{7}$$

$$I(1_{B\cup C}) = p(B) + [1 - p(B)] \cdot \frac{p(C)}{p(A\cup C)} \tag{8}$$

- if $E = (B \cup C)$,

$$I(1_{A\cup B}) = p(A) + [1 - p(A)] \cdot \frac{p(B)}{p(B\cup C)} \tag{9}$$

$$I(1_{A\cup C}) = p(A) + [1 - p(A)] \cdot \frac{p(C)}{p(B\cup C)} \tag{10}$$

Because (6)+(7)+(10)=(5)+(8)+(9), we have

$$[1-p(B)] \cdot \frac{p(A)-p(C)}{p(A\cup C)} + [1-p(A)] \cdot \frac{p(C)-p(B)}{p(B\cup C)} + [1-p(C)] \cdot \frac{p(B)-p(A)}{p(A\cup B)} = 0 \tag{11}$$

[2]See Ghirardato et al. (2001), proposition 14.

for any partition of S, hence $p(A)+p(B\cup C)=1$, $p(B)+p(A\cup C)=1$, $p(C)+p(A\cup B)=1$. Then (5) implies $p(A)+p(C)=p(A\cup C)$, (6) implies $p(B)+p(C)=p(B\cup C)$ and (7) implies $p(A)+p(B)=p(A\cup B)$. Therefore $p(A)+p(B)+p(C)=1$ and p is a probability. $\square$

Because MEU and CEU models are monotonic and constant linear representations of $\{\succsim_E\}_{E\in\Sigma}$, the main implication of this result is that such criteria cannot simultaneously satisfy axioms 1 and 2. Otherwise they degenerate into expected utility. Then a way to preserve ambiguity and ambiguity attitudes is to weaken RCA.

Axiom 3 *(Weak Reduction of Compound Acts) There exists E in Σ such that for all f in $\mathcal{A}$ and x and y in X, if $f\sim_E x$ and $f\sim_{E^c} y$ then $f\sim x_E y$.*

It implies RCA only for a subset of events. We will see in the next section that this weakening may be useful in order to avoid additive beliefs.

Another way to avoid expected utility is to do not assume the consequentialism axiom. However, the updating rules axiomatized for NEU preferences assume this axiom[3]. Precisely, consequentialism imposes two distinct requirements:

1. The updating rule gives no likelihood to events outside of E;
2. The updating rule is unique, in the sense that the DM always uses the same updating rule with no regards for counterfactual outcomes.

Point 1 can be seen as a basic normative requirement. Moreover, if it is not assumed, then a dynamic decision problem cannot, in general, be resolved by backward induction. However, we will see that even if point 2 is not satisfied, backward induction can be used in several situations.

Our approach allows to compute $I_E(1_A)$ when $A\subset E$ or when $E^c\subset A$ for the two NEU criteria considered in this paper -MEU and CEU. Therefore, under RCA, *even if* consequentialism does not universally hold, there are situations where conditional preferences are well-defined. As illustrated in the example below, it allows to describe Ellsberg-type preferences whereas the DM's preferences have a recursive structure.

Example 1 Consider the following dynamic extension of the Ellsberg paradox, as proposed by Epstein and Schneider (2003). A MEU decision maker is facing an urn with 30 red balls and 60 blue or green balls. At time 1, a ball is drawn and the decision maker knows whether this ball is green or not. At time 2, the color of this ball is fully revealed to the decision maker. The state space is $S=\{R,B,G\}$, where R,B and G have obvious signification, and conditional preferences are $\{\succsim_{R\cup B},\succsim_G\}$. A possible set of priors may be:

$$\mathcal{P}=\left\{p=\left(\frac{1}{3},\beta,\frac{2}{3}-\beta\right)|\beta\in\left[\frac{1}{6};\frac{1}{2}\right]\right\} \tag{12}$$

Several bets, that are maps from S to $X=\{0,1\}$, are proposed to the DM. Ellsberg-type preferences are $(1,0,0)\succ(0,1,0)$ and $(0,1,1)\succ(1,0,1)$, where any bet can be described as $(x,R;y,B;z,G)$, with $x,y,z\in X$. Assume that the utility $u:X\to R$ is

[3]See for instance Gilboa and Schmeidler (1993), Pires (2002), Wang (2003) and Eichberger et al. (2007).

linear. Then, by proposition 4, if axiom 2 and 3 (with $E = R \cup B$) hold, then the DM considers the rectangular set of priors $\mathcal{Q}$ such that:

$$\mathcal{Q} = \left\{ q = \left(\frac{1}{3} \frac{\frac{1}{3} + \beta'}{\frac{1}{3} + \beta}, \beta \frac{\frac{1}{3} + \beta'}{\frac{1}{3} + \beta}, \frac{2}{3} - \beta' \right) \mid \beta, \beta' \in \left[\frac{1}{6}, \frac{1}{2} \right] \right\} \tag{13}$$

As noted by Epstein and Schneider (2003), $\mathcal{Q}$ is not consistent with Ellsberg-type preferences. That is why to simultaneously keep reduction of compound acts and consequentialism may be problematic, even if the former is weakened. Therefore, we propose to relax consequentialism. Then, by proposition 5, axiom 1 delivers an updating rule yielding the following conditional preferences:

$$\min_{p_{R\cup B} \in \mathcal{P}_{R\cup B}} \int_S (1,0,0) dp_{R\cup B} = \frac{2}{3} > \frac{1}{3} = \min_{p_{R\cup B} \in \mathcal{P}_{R\cup B}} \int_S (0,1,0) dp_{R\cup B} \tag{14}$$

$$\min_{p_{R\cup B} \in \mathcal{P}_{R\cup B}} \int_S (1,0,1) dp_{R\cup B} = \frac{2}{5} < \frac{3}{5} = \min_{p_{R\cup B} \in \mathcal{P}_{R\cup B}} \int_S (0,1,1) dp_{R\cup B} \tag{15}$$

according to ex-ante preferences.

3 Choquet Expected Utility

An important class of NEU models is the CEU one. In this model, the beliefs are represented by a Choquet capacity, i.e. a set function $\nu : \Sigma \rightarrow [0,1]$ such that $\nu(\emptyset) = 0$, $\nu(S) = 1$ and $\forall A, B \in \Sigma, A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$. Wakker (1989) and Ghirardato et al. (2001, 2003) axiomatize the Choquet Expected utility representation in a Savage framework.

Definition 1 *(CEU) The preference relation is represented by a Choquet Expected Utility functional $I : \mathcal{A} \longrightarrow R$ if there exist a unique capacity ν and a real-valued function $u : X \rightarrow R$, unique up to a positive affine transformation, s.t. the value of any act f is given by:*

$$I : f \longmapsto \int_S u[f(s)] d\nu(s)$$

Therefore, $\{\succsim_E\}_{E\in\Sigma}$ satisfy CEU if and only if each preference relation $\succsim_E$ is represented by the Choquet integral of utility $\int_S u(f) d\nu_E$, where ν_E denotes the conditional set function for $\nu(.)$ given E. Finally, note that the Choquet expectation of 1_A allows to define the capacity of event A: $I(1_A) = \nu(A)$ and $I_E(1_A) = \nu_E(A)$ for all A and E in Σ.

The main implication of the proposition 1 is that we have to make a choice between consequentialism and reduction of compound acts when assuming that $\{\succsim_E\}_{E\in\Sigma}$ are represented by the CEU model. A current way in the literature is to release RCA in WRCA.

Proposition 2 *Let $\{\succsim_E\}_{E\in\Sigma}$ be a class of preference relations on $\mathcal{A}$. Then the following two statements are equivalent:*

i $\{\succcurlyeq_E\}_{E\in\Sigma}$ *satisfy CEU and axioms 2 and 3;*

ii There exist E *and* E^c *in* Σ *and a unique probability measure* $p : \{E, E^c\} \longrightarrow [0;1]$ *such that the preference relation* $\succcurlyeq$ *is represented by:*

$$I : f \longmapsto p(E) \cdot \int_S u(f) d\nu_E + p(E^c) \cdot \int_S u(f) d\nu_{E^c} \tag{16}$$

and, moreover,

$$\forall A \in \Sigma, \nu_B(A) = \frac{\nu(A \cap B)}{p(B)} \tag{17}$$

where $B \in \{E, E^c\}$.

Proof The implication $(ii) \Longrightarrow (i)$ is straightforward. We prove $(i) \Longrightarrow (ii)$. Let $S = (A \cup B \cup C)$, with $A \cap B = B \cap C = A \cap C = \emptyset$, $E = A \cup B$, and $x, y, z \in X$ such that $x > y > z$. Let $x_A z \sim y_E z$, hence $\int_S u(x_A z) d\nu = \int_S u(y_E z) d\nu$ under CEU. Then,

$$u(z) + [u(x) - u(z)]\nu(A) = u(z) + [u(y) - u(z)]\nu(E) \tag{18}$$

if and only if, by axioms 2 and 3,

$$I\left[\int_E u(x_A z) d\nu_E), \int_{E^c} u(x_A z) d\nu_{E^c}\right] = I\left[\int_E u(y_E z) d\nu_E), \int_{E^c} u(y_E z) d\nu_{E^c}\right] \tag{19}$$

if and only if

$$I\left[\int_E u(x_{A\cup C} z) d\nu_E), \int_{E^c} u(x_{A\cup C} z) d\nu_{E^c}\right] = I\left[\int_E u(y_E x) d\nu_E), \int_{E^c} u(y_E x) d\nu_{E^c}\right] \tag{20}$$

that implies $\int_S u(x_{A\cup C} z) d\nu = \int_S u(y_E x) d\nu$. Then,

$$u(z) + [u(x) - u(z)]\nu(A \cup C) = u(y) + [u(x) - u(y)]\nu(C) \tag{21}$$

and, moreover, we have

$$I\left[\int_E u(x_A z_B y) d\nu_E), \int_{E^c} u(x_A z_B y) d\nu_{E^c}\right] = u(y) \tag{22}$$

hence

$$u(z) + [u(y) - u(z)]\nu(A \cup C) + [u(x) - u(y)]\nu(A) = u(y) \tag{23}$$

W.l.o.g. we normalize the utility function $u(.)$ such that $u(x) = 1$ et $u(z) = 0$. Equation (18) yields

$$u(y) = \frac{\nu(A)}{\nu(E)} \tag{24}$$

Equation (21) yields

$$u(y) = \frac{\nu(A \cup C) - \nu(C)}{1 - \nu(C)} \tag{25}$$

and equation (23) yields

$$u(y) = \frac{\nu(A)}{1 + \nu(A) - \nu(A \cup C)} \tag{26}$$

Consequently, by equalizing these last equations,

$$\frac{1+\nu(A)-\nu(E)-\nu(E^c)}{1-\nu(E^c)}=\frac{\nu(A)}{\nu(E)} \tag{27}$$

and, if ν not additive on $\{E, E^c\}$, then $\exists\varepsilon \in \mathbb{R}\backslash\{0\}$ such that:

$$\frac{\nu(A)-\varepsilon}{\nu(E)-\varepsilon}=\frac{\nu(A)}{\nu(E)} \tag{28}$$

If $\nu(A) \neq \nu(E)$, then $\varepsilon = 0$ which is a contradiction, hence ν is additive on $\{E, E^c\}$.

Moreover, because by equation (18) $u(y) = \nu_E(A)$ when $A \subset E = A \cup B$, axiom 2 implies

$$\forall A \in \Sigma, \nu_E(A) = \frac{\nu(A \cap E)}{p(E)} \tag{29}$$

and the same implication holds on E^c. □

Proposition 2 states that when RCA is weakened, the representation I is additive only on the subset $\{x_E y \in X^2 | \exists f \in \mathcal{A} : f \sim_E x, f \sim_{E^c} y\}$ of binary acts. Sarin and Wakker (1998) found a similar result with different assumptions: they assumed dynamic consistency, consequentialism and their sequential consistency property.

Another way to avoid expected utility is to do not assume consequentialism. However, given any event E in Σ, all updating rules commonly used to condition Choquet capacities imply $\nu_{E^c}(A) = 0$ when $A \subset E^c$. Then a relevant question may be: How define conditional capacities if we do not explicitly assume that counterfactual outcomes do not matter ? We give a partial answer to this question by characterizing the conditional capacity in several situations.

Proposition 3 *Let $\{\succsim_E\}_{E\in\Sigma}$ be a class of preference relations on $\mathcal{A}$. If $\{\succsim_E\}_{E\in\Sigma}$ satisfy CEU and axiom 1, then for all A, E in Σ:*

i If $A \subset E$, then

$$\nu_E(A) = \frac{\nu(A \cap E)}{\nu(E)} \tag{30}$$

ii If $E \subset A$, then

$$\nu_{E^c}(A) = \frac{\nu((A \cap E^c) \cup E) - \nu(E)}{1-\nu(E)} \tag{31}$$

iii If $E^c \subset A$, then

$$\nu_E(A) = \frac{\nu((A \cap E) \cup E^c) - \nu(E^c)}{1-\nu(E^c)} \tag{32}$$

iv If $A \subset E^c$, then

$$\nu_{E^c}(A) = \frac{\nu(A \cap E^c)}{\nu(E^c)} \tag{33}$$

Proof Applied to the characteristic function of any event A, CEU and axiom 1 yield

$$\int_S 1_A d\nu = I\left[\int_S 1_A d\nu_E, \int_S 1_A d\nu_{E^c}\right] \tag{34}$$

where $\int_S 1_A d\nu = \nu(A)$, $\int_S 1_A d\nu_E = \nu_E(A)$ and $\int_S 1_A d\nu_{E^c} = \nu_{E^c}(A)$. Then,

(i) $A \subset E \Rightarrow \nu_E(A) \geq \nu_{E^c}(A) = 0$ hence $\nu(A) = \nu_E(A)\nu(E)$.

(ii) $E \subset A \Rightarrow \nu_E(A) = 1 \geq \nu_{E^c}(A)$ hence $\nu(A) = \nu_{E^c}(A) + [1 - \nu_{E^c}(A)]\nu(E)$.

(iii) $E^c \subset A \Rightarrow \nu_{E^c}(A) = 1 \geq \nu_E(A)$ hence $\nu(A) = \nu_E(A) + [1 - \nu_E(A)]\nu(E^c)$.

(iv) $A \subset E^c \Rightarrow \nu_{E^c}(A) \geq \nu_E(A) = 0$ hence $\nu(A) = \nu_{E^c}(A)\nu(E^c)$.

□

Therefore, the recursive relation allowed by RCA defines an updating rule for ν. Given any event E in Σ, it consists to use the Bayes' rule when $A \subset E$ and the Dempster-Shafer's rule when $E^c \subset A$ to calculate ν_E. These rules have been previously axiomatized by Gilboa and Schmeidler (1993) for the case where ν is convex (see also Wang, 2003, for the general case).

To see that this updating rule can violate consequentialism, let $S = (A \cup B \cup C)$, $A \cap B = B \cap C = A \cap C = \emptyset$, $E = (A \cup B)$, $f = x_A y_B z$ and $g = x_A y_B z'$, with $z' \geq x \geq y \geq z$. Then the conditional capacity in $\int u(f) d\nu_E$ is given by statement (i) whereas the conditional capacity in $\int u(g) d\nu_E$ is given by statement (iii). Therefore, $I_E[u(f)]$ and $I_E[u(g)]$ differ in general and axiom 2 fails to hold.

4 Maximin Expected Utility

In this section, we suppose that the DM considers a non-empty, compact and convex set $\mathcal{P}$ of finitely additive probability measures, and maximizes expected utility with respect to the lower probability. MEU over Savage acts is axiomatized in Ghirardato et al. (2001, 2003)[4].

Definition 2 *(MEU) The preference relation is represented by a Maximin Expected Utility functional $I : \mathcal{A} \longrightarrow R$ if there exist a non-empty, compact and convex set $\mathcal{P}$ of finitely additive probability measures on Σ and a real-valued function $u : X \to R$, unique up to a positive affine transformation, s.t. the value of any f is given by:*

$$I : f \longmapsto \min_{p \in \mathcal{P}} \int_S u[f(s)] dp(s)$$

Moreover, given an event E, the conditional MEU of f, noted $\min_{p_E \in \mathcal{P}_E} \int_S u(f) dp_E$, uses the set of conditional probabilities $\mathcal{P}_E$.

Again, a way to avoid expected utility is to weaken axiom 1 into axiom 3. The following result states that a necessary and sufficient condition to weak reduction of

[4]Other axiomatizations are Casadesus-Masanell et al. (2000) and Alon and Schmeidler (2009).

compound acts and consequentialism within the multiple priors framework is the rectangularity of the set of priors. Because the former implies backward induction on a given and fixed filtration, it is convenient to distinguish the restriction of $\mathcal{P}$ to $\{E, E^c\}$ and we note it $\mathcal{P}(E, E^c) = \{m \in \mathcal{P} | m : \{E, E^c\} \longrightarrow]0; 1[\}$.

Proposition 4 *Let $\{\succsim_E\}_{E\in\Sigma}$ be a class of preference relations on $\mathcal{A}$. Then the following two statements are equivalent:*

i $\{\succsim_E\}_{E\in\Sigma}$ satisfy MEU and axioms 2 and 3;

ii There exist E and E^c in Σ and a compact, convex and non-empty set of priors $\mathcal{P}$ such that

$$\mathcal{P}(A) = \left\{ \int_S p_B(A) dm | A \subset B \in \{E, E^c\}, p_B \in \mathcal{P}_B, m \in \mathcal{P}(E, E^c) \right\} \tag{35}$$

and $\succsim$ is represented by

$$I : f \longmapsto \min_{m\in\mathcal{P}(E,E^c)} \int_S \left(\min_{p_E\in\mathcal{P}_E} \int_S u(f) dp_E, \min_{p_{E^c}\in\mathcal{P}_{E^c}} \int_S u(f) dp_{E^c} \right) dm \tag{36}$$

where $u : X \longrightarrow R$ is unique up to a positive affine transformation.

Proof $(ii) \Rightarrow (i)$ is straightforward. We prove $(i) \Rightarrow (ii)$. For all $A \subset B \in \{E, E^c\}$, MEU and axioms 2 and 3 imply:

$$\min_{p\in\mathcal{P}} \int_S 1_A dp = \min_{m\in\mathcal{P}(E,E^c)} m(B) \cdot \min_{p_B\in\mathcal{P}_B} \int_S 1_A dp_B \tag{37}$$

hence $p_*(A) = \min_{m\in\mathcal{P}(E,E^c)} m(B) \cdot \min_{p_B\in\mathcal{P}_B} p_B(A)$, with $p_* \in \arg\min_{p\in\mathcal{P}} \int_S 1_A dp$. Let $p^* \in \arg\min_{p\in\mathcal{P}} \int_S 1_{A^c} dp$. Because $p^*(A) = \max_{p\in\mathcal{P}} p(A) = 1 - I[\min \int_E 1_{A^c} dp_E, \min \int_{E^c} 1_{A^c} dp_{E^c}] = 1 - I[1 - \max_{p_B\in\mathcal{P}_B} p_B(A), 1] = 1 - \{ \max_{m\in\mathcal{P}(E,E^c)} m(B)[1 - \max_{p_B\in\mathcal{P}_B} p_B(A)] + 1 - \max_{m\in\mathcal{P}(E,E^c)} m(B)\} = \max_{m\in\mathcal{P}(E,E^c)} m(B) \cdot \max_{p_B\in\mathcal{P}_B} p_B(A)$,

$$p_*(A) \leq m(B) \frac{p(A)}{p(B)} \leq p^*(A) \tag{38}$$

for any m in $\mathcal{P}(E, E^c)$ and any p_B in $\mathcal{P}_B$. Then, $\exists \alpha \in [0, 1]$ such that

$$\alpha p_*(A) + (1 - \alpha) p^*(A) = m(B) \frac{p(A)}{p(B)} \tag{39}$$

By convexity of $\mathcal{P}$, $\mathcal{P}(A) = \{\alpha p_*(A) + (1 - \alpha) p^*(A) | \alpha \in [0; 1]\}$ and then, by convexity of $\mathcal{P}(E, E^c)$ and $\mathcal{P}_B$,

$$\mathcal{P}(A) = \left\{ \int_S p_B(A) dm | A \subset B \in \{E, E^c\}, p_B \in \mathcal{P}_B, m \in \mathcal{P}(E, E^c) \right\} \tag{40}$$

$\square$

Statement (ii) in the proposition says that the set $\mathcal{P}$ is rectangular. Such a condition has been introduced by Epstein and Schneider (2003), and the reader is referred to their article for an extensive discussion of this concept.

Symmetrically to the CEU model, if one wish to keep axiom 1, then we cannot assume consequentialism. In the MEU framework, this means that we have to update such and such subset of $\mathcal{P}$, depending on counterfactual outcomes.

Proposition 5 *Let $\{\succsim_E\}_{E\in\Sigma}$ be a class of preference relations on $\mathcal{A}$. If $\{\succsim_E\}_{E\in\Sigma}$ satisfy MEU and axiom 1, then for all A, E in Σ:*

i If $A \subset E$, then

$$\mathcal{P}_E(A) = \left\{ p_E(A) = \frac{p(A\cap E)}{p(E)} | p \in \arg\min_{p\in\mathcal{P}} p(E) \right\} \tag{41}$$

ii If $E \subset A$, then

$$\mathcal{P}_{E^c}(A) = \left\{ p_{E^c}(A) = \frac{p(A\cap E^c)}{p(E^c)} | p \in \arg\max_{p\in\mathcal{P}} p(E^c) \right\} \tag{42}$$

iii If $E^c \subset A$, then

$$\mathcal{P}_E(A) = \left\{ p_E(A) = \frac{p(A\cap E)}{p(E)} | p \in \arg\max_{p\in\mathcal{P}} p(E) \right\} \tag{43}$$

iv If $A \subset E^c$, then

$$\mathcal{P}_{E^c}(A) = \left\{ p_{E^c}(A) = \frac{p(A\cap E^c)}{p(E^c)} | p \in \arg\min_{p\in\mathcal{P}} p(E^c) \right\} \tag{44}$$

Proof Applied to the characteristic function of any event A, MEU and axiom 1 yield

$$\min_{p\in\mathcal{P}} \int_S 1_A dp = I\left[\min_{p_E\in\mathcal{P}_E} \int_S 1_A dp_E, \min_{p_{E^c}\in\mathcal{P}_{E^c}} \int_S 1_A dp_{E^c}\right] \tag{45}$$

where $\min\limits_{p\in\mathcal{P}} \int_S 1_A dp = \min\limits_{p\in\mathcal{P}} p(A)$, $\min\limits_{p_E\in\mathcal{P}_E} \int_S 1_A dp_E = \min\limits_{p_E\in\mathcal{P}_E} p_E(A)$ and $\min\limits_{p_{E^c}\in\mathcal{P}_{E^c}} \int_S 1_A dp_{E^c} = \min\limits_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A)$. Then,

(i) Let $p \in \arg\min\limits_{p\in\mathcal{P}} \int_S 1_A dp$. $A \subset E \Rightarrow \min\limits_{p_E\in\mathcal{P}_E} p_E(A) \geq \min\limits_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A) = 0$ hence $p(A) = \min\limits_{p_E\in\mathcal{P}_E} p_E(A) \cdot \min\limits_{p\in\mathcal{P}} p(E)$. Moreover, $p(A) = p_E(A)p(E)$. If $p(E) > \min\limits_{p\in\mathcal{P}} p(E)$, then $p_E(A) < \min\limits_{p_E\in\mathcal{P}_E} p_E(A)$ which is a contradiction. Therefore, $p \in \arg\min\limits_{p\in\mathcal{P}} \int_S 1_A dp$ $\bigcap \arg\min\limits_{p\in\mathcal{P}} \int_S 1_E dp$ and then

$$\min_{p_E\in\mathcal{P}_E} p_E(A) = \frac{p(A\cap E)}{p(E)} \tag{46}$$

(ii) Let $p \in \arg\min_{p\in\mathcal{P}} \int_S 1_A dp$. $E \subset A \Rightarrow \min_{p_E\in\mathcal{P}_E} p_E(A) = 1 \geq \min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A)$ hence $\min_{p\in\mathcal{P}} p(A) = \min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A) + [1 - \min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A)]\min_{p\in\mathcal{P}} p(E) = \min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A)[1 - \min_{p\in\mathcal{P}} p(E)] + \min_{p\in\mathcal{P}} p(E)$. Moreover, $p(A) = p_{E^c}(A)p(E^c) + p(E)$ and $0 \leq p_{E^c}(A) \leq 1$. If $p(E) > \min_{p\in\mathcal{P}} p(E)$, then $p(A) > p_{E^c}(A)[1 - \min_{p\in\mathcal{P}} p(E)] + \min_{p\in\mathcal{P}} p(E) \geq p(A)$ which is a contradiction. Therefore, $p \in \arg\min_{p\in\mathcal{P}} \int_S 1_A dp \bigcap \arg\min_{p\in\mathcal{P}} \int_S 1_E dp$ and then

$$\min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A) = \frac{p(A\cap E^c)}{p(E^c)} \tag{47}$$

(iii) Let $p \in \arg\min_{p\in\mathcal{P}} \int_S 1_A dp$. $E^c \subset A \Rightarrow \min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A) = 1 \geq \min_{p_E\in\mathcal{P}_E} p_E(A)$ hence $\min_{p\in\mathcal{P}} p(A) = \min_{p_E\in\mathcal{P}_E} p_E(A) + [1 - \min_{p_E\in\mathcal{P}_E} p_E(A)]\min_{p\in\mathcal{P}} p(E^c)$. The same argument that in the previous case implies $p \in \arg\min_{p\in\mathcal{P}} \int_S 1_A dp \bigcap \arg\min_{p\in\mathcal{P}} \int_S 1_{E^c} dp$.

(iv) Let $p \in \arg\min_{p\in\mathcal{P}} \int_S 1_A dp$. $A \subset E^c \Rightarrow \min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A) \geq \min_{p_E\in\mathcal{P}_E} p_E(A) = 0$ hence $\min_{p\in\mathcal{P}} p(A) = \min_{p_{E^c}\in\mathcal{P}_{E^c}} p_{E^c}(A)\min_{p\in\mathcal{P}} p(E^c)$. The same argument that in the case (i) implies $p \in \arg\min_{p\in\mathcal{P}} \int_S 1_A dp \bigcap \arg\min_{p\in\mathcal{P}} \int_S 1_{E^c} dp$.

$\square$

Therefore, axiom 1 allows us to obtain an updating rule for MEU preferences. Note that statements (ii) and (iii) of the proposition correspond to the use of the maximum likelihood procedure applied to $\mathcal{P}$ (see Gilboa and Schmeidler, 1993). Again, such a way of updating is not consequentialist in general. Furthermore, it coincides with the updating rule obtained in proposition 3 for CEU preferences when there exists a convex capacity (see Chateauneuf et al., 2001).

5 Conclusion

We have shown that if we assume reduction of compound acts and consequentialism together with a monotonic constant linear representation, then we obtain the classical EU model from Savage (1954) with additive beliefs. The novelty of our approach is that we explicitly assume a reduction of compound acts axiom. Its weakening allows us to reviewed some existing results and to avoid expected utility. Notably, concerning MEU preferences, the set of priors must be rectangular.

However, when the DM's preferences exhibit typical violations of the Savage's Sure-thing principle, the recursive structure of the preferences can be maintained in several situations if we drop consequentialism, in the sense that conditional preferences are depending on counterfactuals outcomes. Further, Reduction of compound acts has not to be weaken. It can be suitable for economic applications when, for instance, one wish to compare dynamic situations where uncertainty is differently resolved. Then we axiomatize an updating rule for CEU and MEU preferences allowing recursion. In our knowledge, such a way of updating is the only one to be dynamically consistent but non-consequentialist in the CEU framework. Concerning MEU preferences, Hanany and Klibanoff (2007) also propose a non-consequentialist updating rule. However, in their setting, this means that ex-ante preferences determine conditional preferences by selecting such and such subset of the set of priors when conditioning. They assume a weakened form of dynamic consistency that does not allow recursion.

In our agenda for future search, we consider a behavioral interpretation of the updating rules deduced in this paper.

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